The W_N minimal model classification

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Abstract. We first rigourously establish, for any $N \geq 2$, that the toroidal modular invariant partition functions for the (not necessarily unitary) $W_N(p,q)$ minimal models *biject* onto a well-defined subset of those of the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ Wess-Zumino-Witten theories at level $(p-N, q-N)$. This permits considerable simplifications to the proof of the Cappelli-Itzykson-Zuber classification of Virasoro minimal models. More important, we obtain from this the complete classification of all modular invariants for the $W_3(p,q)$ minimal models. All should be realised by rational conformal field theories. Previously, only those for the unitary models, i.e. $W_3(p,p+1)$, were classified. For all N our correspondence yields for free an extensive list of $W_N(p,q)$ modular invariants. The W_3 modular invariants, like the Virasoro minimal models, all factorise into SU(3) modular invariants, but this fails in general for larger N. We also classify the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ modular invariants, and find there a new infinite series of exceptionals.

1 Introduction

Each chiral half of a rational conformal field theory (RCFT) is controlled by a chiral algebra (rational vertex operator algebra). These associate to each surface a finite-dimensional space of conformal blocks, possessing appropriate conditions of analyticity, factorisation, monodromy etc.

The quantities in the full RCFT of greatest interest are certain sesquilinear combinations of conformal blocks called correlation functions. A beautiful theory (see e.g. [28]), related at least in broad strokes to the subfactor approach to RCFT (see e.g. [4]), finds these sesquilinear combinations starting from a special symmetric Frobenius algebra of the category of modules of the chiral algebra. This theory regards the 1-loop open string partition function (*nim-rep*) as more fundamental than the 1-loop closed string partition function (*modular invariant*). However, the latter is far more tightly constrained and within this framework the classification of RCFT associated to a given chiral algebra would (except for the easiest examples) proceed first by classifying the modular invariants, then the corresponding nim-reps, and then from that the possible Frobenius

algebras. For example, if the modular invariant is a permutation matrix, then the corresponding Frobenius algebra is Azumaya. More precisely, the baby case, namely SU(2) Wess-Zumino-Witten (WZW), is the exception as its nim-reps correspond to graphs with largest eigenvalue < 2 and so are easily classified. But already SU(3) WZW nim-reps seem hopeless to classify and include infinitely many 'nonphysical' nim-reps (the first at level 3, given by quantum-dimension). 'Non-physical' here means nim-reps not corresponding to a modular invariant. The point is that, even in this categorical framework, the starting point for the RCFT classification in practice would be the modular invariant classification.

The Virasoro minimal models constitute perhaps the best known RCFTs. They belong to a sequence of reasonably accessible rational chiral algebras: the $W_N(p,q)$ minimal models for $N \geq 2$, generated by fields of conformal weight $2, 3, \ldots, N$. Here p, q are *coprime* integers, i.e. $\gcd(p,q) = 1$. W_2 recovers the Virasoro algebra, while W_3 was introduced by A. B. Zamolodchikov [29], and the generalisation to higher N is due to Fateev-Lukyanov [11] (see also [6, 7]). The minimal models $W_N(p,q)$ are related for instance to fractional level admissible modules of affine A_{N-1} (see e.g. [12, 26]). For instance, $W_N(p,q)$ is realised by the Goddard-Kent-Olive diagonal coset $\mathrm{su}(N)_k \oplus \mathrm{su}(N)_1/\mathrm{su}(N)_{k+1}$ where the level k is $p/|q-p| - N$.

The modules of a rational chiral algebra naturally inherit a Hermitian inner product. An RCFT is called *unitary* if this inner product is positive-definite. For example the WZW models (i.e. the RCFT associated to compact groups and affine algebras at integral level) are unitary, while the W_N minimal models are typically non-unitary. Most attention in the literature has focussed on unitary RCFT, but this seems unwarranted. For example, in the statistical mechanical realisation of RCFTs unitarity has no physical significance — for example, 2-dimensional ferromagnets at criticality are described by the non-unitary Yang–Lee model, $W_2(2,5)$. Moreover, in string theory the matter CFT is coupled to the non-unitary (super-)ghost CFT; what must be unitary is the corresponding BRST cohomology, but the relationship between the (non)unitarity of the matter CFT and that of BRST cohomology isn't obvious. Also, very little marks the structural distinction between unitary and non-unitary vertex operator algebras.

However, from the point of view of modular invariant classifications, non-unitarity *does* introduce significant additional challenges, ultimately because the vacuum is not the primary of minimal conformal weight. Addressing a large nontrivial class of these is the main motivation for this paper. (The only previous non-unitary classifications were $W_2(p,q)$ and SU(2) at fractional level, and these could avoid these aforementioned serious challenges through a technicality unavailable for $N > 2$, as explained in Section 3.3.)

In particular, non-unitary $W_N(p,q)$ looks more akin to WZW at *fractional* than *integral* level (recall the coset description given above). Modular invariant classifications for fractional level WZW are notoriously wild (see [25] for the SU(2) story). Is the tameness of the 'A-D-E' classification [9, 8] for W_2 a low rank accident?

More precisely, the $W_N(p,q)$ *modular data* (i.e. the matrices S and T defining the modular group representation) looks like that of $\mathrm{SU}(N) \times \mathrm{SU}(N)$ WZW modular data at fractional heights p/q and q/p (height$=$ $N+$level). Applying to this modular data the *Galois shuffle* of [17], we

associate to each $W_N(p,q)$ minimal model a unique $\mathrm{SU}(N)\times\mathrm{SU}(N)$ modular invariant at *integral* heights p,q. For $N=2$ and $N=3$, the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ modular invariants corresponding to W_N minimal models can always be expressed in terms of those of $\mathrm{SU}(N)$ together with simple-current modular invariants (though the proof for $N=3$ is difficult, as we see in this paper). Even if this pattern were to continue for higher N, the complete list of $\mathrm{SU}(N)$ WZW modular invariants is not known for $N>3$, making the W_N classification for higher N out of reach for now.

The $W_2(p,q)$ minimal model classification (both unitary and non-unitary) is due to Cappelli–Itzykson–Zuber [9]. Our $W_N \hookrightarrow$ WZW relation permits significant simplifications to this proof. The *unitary* W_3 minimal models, i.e. $W_3(p,p+1)$, were found in [18] using the coset realisation $\mathrm{su}(3)_{p-3}\oplus\mathrm{su}(3)_1/\mathrm{su}(3)_{p-2}$. In this paper, we use the $W_N \hookrightarrow$ WZW relation to obtain the full $W_3(p,q)$ classification (i.e. non-unitary as well as unitary). This project (obtaining the W_3 minimal model classification via the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ one) constituted parts of the first author's theses [2, 3]. As was the case for W_2, *all of the W_3 modular invariants (but only about half of the* $\mathrm{SU}(3)\times\mathrm{SU}(3)$ *ones) come in factorised form.* All W_3 modular invariants have a corresponding nim-rep and we expect will arise as the partition function of a healthy RCFT.

This is not true for $\mathrm{SU}(3)\times\mathrm{SU}(3)$. In particular we find a new infinite series of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ exceptional modular invariants, the first at height $(12,5)$:

$$\begin{aligned}|\chi_{[11](11)}+\chi_{[55](11)}|^2+|\chi_{[11](22)}+\chi_{[55](22)}|^2+|\chi_{[33](31)}+\chi_{[33](13)}|^2+|\chi_{[33](12)}+\chi_{[33](21)}|^2\\ +\chi_{[33](11)}\big(\chi_{[11](31)}+\chi_{[55](31)}+\chi_{[11](13)}+\chi_{[55](13)}\big)^*+c.c.\\ +\chi_{[33](22)}\big(\chi_{[11](21)}+\chi_{[55](21)}+\chi_{[11](12)}+\chi_{[55](12)}\big)^*+c.c.\ ,\end{aligned} \tag{1.1}$$

where we use the shorthand $\chi_{[ab](cd)}$ for the affine $A_2\oplus A_2$ character combination

$$\chi_{(12-a-b;a-1,b-1),(5-c-d;c-1,d-1)}+\chi_{(b-1;12-a-b,a-1),(5-c-d;c-1,d-1)}+\chi_{(a-1;b-1,12-a-b),(5-c-d;c-1,d-1)}\,.$$

None of these new exceptionals can be partition functions of an RCFT. The reason is that any such partition function must be a twist of one of extension type [27], i.e. a sum of squares, and no such 'extension type' modular invariant exists for these exceptionals.

An interesting extension of the present work would be to extend our W-algebra $\hookrightarrow$ WZW relation to the W-algebras corresponding to any simple Lie algebra. Explicit classifications won't be possible for now, except for small level.

We begin with a review of modular invariants (Section 2.1) and $\mathrm{SU}(N)$ modular data (Section 2.2). Section 3 is the heart of the paper. It reduces the $W_N(p,q)$ minimal model classification to that of $\mathrm{SU}(N)\times\mathrm{SU}(N)$ at coprime heights, and states the W_3 and $\mathrm{SU}(3)\times\mathrm{SU}(3)$ classifications. Sections 4 and 5 prove the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ (and hence W_3) modular invariant classification. The main results of this paper are Theorem 3.3 (giving the W_3 minimal model classification) and Theorem 3.1 (explicitly relating W_N to $\mathrm{SU}(N)\times\mathrm{SU}(N)$). Also of independent interest should be Theorem 3.2 (giving the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ classification) and Section 3.3 (sketching a new proof for the Virasoro minimal models).

2 Background

For a review of much of the basics of RCFT, see e.g. [10], and for necessary aspects of modular data and modular invariants, see [20]. The subtleties occurring in non-unitary theories are discussed in [17]. The standard references on W-algebras are the review [6] and the reprint volume [7].

2.1 Modular invariants

The spectrum of an RCFT can be read off from its 1-loop torus partition function:

$$\mathcal{Z} = \sum_{\lambda,\mu\in\Phi} M_{\lambda\mu}\chi_\lambda\chi_\mu^* \ , \tag{2.1}$$

where λ, μ run over the (finitely many) chiral primaries, parametrising the irreducible modules of the chiral algebra of the theory. Let 0 denote the vacuum sector. The functions χ_λ are conformal blocks for the torus, and characters of those modules. Throughout this paper, $*$ denotes complex conjugation. The χ_λ realise a unitary representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$, via

$$\chi_\lambda\left(\frac{-1}{\tau}\right) = \sum_{\nu\in\Phi} S_{\lambda\nu}\chi_\nu(\tau) \ , \qquad \chi_\lambda(\tau+1) = \sum_{\nu\in\Phi} T_{\lambda\nu}\chi_\nu(\tau) \ . \tag{2.2}$$

The partition function (2.1) must be invariant under this action of $\mathrm{SL}_2(\mathbb{Z})$.

The coefficient matrix M in (2.1) is called a *modular invariant*. More formally, we call M a *modular invariant* if the following three conditions hold:

$$\begin{aligned} M_{00} &= 1 && (\textit{uniqueness of vacuum}), && (2.3)\\ M_{\lambda\mu} &\in \mathbb{Z}_{\geq 0}\ \forall\lambda,\mu\in\Phi && (\textit{integrality and nonnegativity}), && (2.4)\\ SM &= MS,\ TM = MT && (\textit{modular invariance}). && (2.5) \end{aligned}$$

The matrices S, T in (2.2) are called modular data. T is a diagonal matrix, with entries $T_{\lambda\lambda} = \exp[2\pi\mathrm{i}\,(h_\lambda - \frac{c}{24})]$ where h_λ is the conformal weight and c is the central charge. The matrix S is more subtle but more important. For instance the well-known Verlinde's formula expresses the fusion coefficients in terms of S. Let us focus on S.

S is a symmetric unitary matrix, $S_{\lambda\mu} = S_{\mu\lambda}$, whose square S^2 is an order-2 permutation matrix called *charge-conjugation* C:

$$\begin{aligned} S_{C\lambda,\mu} &= S_{\lambda,C\mu} = S^*_{\lambda\mu} \ , && (2.6)\\ T_{C\lambda,C\mu} &= T_{\lambda\mu} \ . && (2.7) \end{aligned}$$

Note that if M is a modular invariant, so will be the matrix product $CM = MC$. More generally, if M' is a *permutation invariant* (see (2.17) below), then the products MM' and $M'M$ will also be modular invariants (though not necessarily equal).

Simple-currents are permutations J of the primaries $\lambda \in \Phi$, such that

$$S_{J\lambda,\mu} = \exp[2\pi \mathrm{i}\, Q_J(\mu)] S_{\lambda\mu} \ , \tag{2.8}$$

for some rational numbers $Q_J(\mu)$. The primary $J0 \in \Phi$ is also called a simple-current.

The *Galois symmetry* is both the most powerful and the most exotic. All entries $S_{\lambda\mu}$ lie in the cyclotomic field $\mathbb{Q}[\xi_L]$ for some root of unity $\xi_L := \exp[2\pi \mathrm{i}/L]$. The Galois automorphisms σ_ℓ are parametrised by integers $\ell \in \mathbb{Z}_L^\times$ coprime to L, defined mod L. For any such ℓ, there is a permutation of Φ, also labelled σ_ℓ, and a choice of signs $\epsilon_\ell(\lambda) \in \{\pm 1\}$ such that

$$\sigma_\ell(S_{\lambda\mu}) = \epsilon_\ell(\lambda)\, S_{\sigma_\ell\lambda,\mu} = \epsilon_\ell(\mu)\, S_{\lambda,\sigma_\ell\mu} \ . \tag{2.9}$$

For example, $L = 4Nn$ works for $SU(N)$ at level n, and $L = 4Npq$ works for $W_N(p,q)$.

The easiest examples of modular invariants are $M = I$ and $M = C$. Another generic source are simple-currents. Let J be of order d, where the conformal weight h_{J0} lies in $\frac{1}{d}\mathbb{Z}$ (this is automatic if d is odd). Then there is a modular invariant associated to it by

$$M[J]_{\lambda\mu} = \sum_{1 \le j \le d} \delta^{\mathbb{Z}}\left(Q_J(\lambda) - j\, h_{J0}\right)\, \delta_{\mu\,,J^j\lambda} \ , \tag{2.10}$$

where $\delta^{\mathbb{Z}}(x)$ equals 1 or 0 depending on whether or not x is integral.

The condition that M commutes with the diagonal matrix T is the selection rule

$$M_{\lambda\mu} \neq 0 \ \Rightarrow \ T_{\lambda\lambda} = T_{\mu\mu} \ . \tag{2.11}$$

The symmetries of S become symmetries of M. For example, charge-conjugation obeys

$$M_{C\lambda,C\mu} = M_{\lambda\mu} \ . \tag{2.12}$$

More generally, the Galois symmetry (2.9), S-invariance (2.5), and integrality (2.4) together yield

$$M_{\lambda\mu} = \epsilon_\ell(\lambda)\, \epsilon_\ell(\mu)\, M_{\sigma_\ell\lambda,\sigma_\ell\mu} \ . \tag{2.13}$$

If $M_{\lambda\mu} \neq 0$, then (2.13) and nonnegativity (2.4) give the *parity rule*

$$M_{\lambda\mu} \neq 0 \Rightarrow \epsilon_\ell(\lambda) = \epsilon_\ell(\mu) \ . \tag{2.14}$$

By the *minimal primary* o we mean the primary $\lambda \in \Phi$ with minimal conformal weight h_λ. Equivalently, o is the unique primary obeying

$$S_{\lambda o} \geq S_{0o} > 0 \ . \tag{2.15}$$

Equality in (2.15) will hold iff λ is a simple-current. When $o \neq 0$ (i.e. when the vacuum column of S is not strictly positive), we call the modular data *non-unitary.* The modular data of WZW

models all have $o=0$, i.e. are *unitary*, but most W_N minimal models are non-unitary. (To our knowledge it is not yet known for every RCFT that there must be a *unique* primary with minimal conformal weight, but this is true for all $W_N(p,q)$ and all unitary theories.)

The ratios $\mathcal{D}\lambda := S_{\lambda o}/S_{0o}$ are called *quantum-dimensions*. Then (2.15) says $\mathcal{D}\lambda \geq 1$ with equality iff λ is a simple-current. Moreover, $\mathcal{D}\lambda = \mathcal{D}\mu$ if (but not in general iff) $\mu = C^a J\lambda$ for some a and some simple-current J.

Lemma 2.1 *[15] Let M be a modular invariant and J, J' be simple-currents. Suppose $o=0$.*

(a) If $M_{J0,J'0} \neq 0$ then $M_{J\lambda,J'\mu} = M_{\lambda\mu}$ for all $\lambda,\mu \in \Phi$. If moreover $M_{\lambda\mu} \neq 0$, then $Q_J(\lambda) \equiv Q_{J'}(\mu)$ (mod 1).

(b) For each $\lambda \in \Phi$, define $s_L(\lambda) = \sum_\mu M_{\mu 0} S_{\lambda\mu}$. Then $s_L(\lambda) \geq 0$, and $s_L(\lambda) > 0$ iff some $\mu \in \Phi$ has $M_{\lambda\mu} \neq 0$ (similarly for $s_R(\lambda) = \sum_\mu M_{0\mu} S_{\lambda\mu}$).

This lemma is crucial to the 'modern' approach (see Steps 1-3 below) to modular invariant classifications, but fails in general for non-unitary modular data. Because of this, non-unitary modular invariant classifications can look very different (see [25, 17] for dramatic examples) and will in general require new arguments.

Fortunately, in many non-unitary RCFTs, the minimal primary and the vacuum are related in a definite way called the Galois shuffle [17]. When this holds, $M_{oo} = 1$ and Lemma 2.1 remains valid. The starting point for this paper is that the Galois shuffle holds for all W_N minimal models.

WZW modular data for compact groups at integral level $k \in \{1,2,3,\ldots\}$ is well-understood; their primaries $\lambda \in \Phi$ are the level k integrable highest weights for the corresponding affine Kac–Moody algebra. The general method which has evolved over the years for their modular invariant classification follows these basic steps:

Step 1 The vacuum row and column of a modular invariant M are heavily constrained, most significantly by (2.11) and (2.14). In this step we solve those constraints for $\mu = 0$. In practise these constraints (usually) force M to obey the condition

$$M_{0\lambda} \neq 0 \text{ or } M_{\lambda 0} \neq 0 \ \Rightarrow \lambda \text{ is a simple-current} . \tag{2.16}$$

Step 2 Find all M obeying (2.16) — these correspond to simple-current extensions of the chiral algebra. For technical reasons, we first find all M obeying the stronger condition

$$M_{0\lambda} = \delta_{0\lambda} \text{ for all } \lambda \in \Phi . \tag{2.17}$$

Modular invariants M satisfying (2.17) are necessarily permutation matrices: $M_{\lambda\mu} = \delta_{\mu,\pi\lambda}$ for some permutation π of Φ (see e.g. Lemma 2 of [15] for a proof). For that reason, these M are called *permutation invariants*; their importance is that multiplying by them sends modular invariants to (usually different) modular invariants.

Step 3 At small levels, modular invariants that do not obey (2.16) can occur. These exceptional invariants must be classified separately.

2.2 SU(N) modular data and SU(3) modular invariants

In Section 3.1 we describe the $W_N(p,q)$ modular data through that of SU(N). This subsection describes the latter, equivalently the affine algebra $A^{(1)}_{N-1}$, at integral levels k. To make some formulas cleaner, we shift the highest weights by the Weyl vector ρ, use the *height* $n := k+N$ rather than the level k, and omit the redundant extended Dynkin label λ_0. We abbreviate 'SU(N) at height n' with '$\mathrm{su}_{N;n}$'.

The $\mathrm{su}_{N;n}$ primaries can be identified with the set

$$\Phi^n_N := \{\lambda = (\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{Z}^{N-1} : 0 < \lambda_i,\ \lambda_1 + \cdots + \lambda_{N-1} < n\} \ , \tag{2.18}$$

and the vacuum (denoted 0 in Section 2.1) with the Weyl vector $\rho = (1, \dots, 1)$. The T matrix is given by $T^{(N;n)}_{\lambda\lambda} = \alpha \exp[\pi \mathrm{i} \lambda^2/n]$ for some constant α, where $\lambda^2 := \lambda \cdot \lambda$ and

$$\lambda \cdot \mu := \sum_{1\le i<N} \frac{i\,(N-i)}{N}\, \lambda_i \mu_i + \sum_{1\le i<j<N} \frac{i\,(N-j)}{N}\, (\lambda_i\mu_j + \lambda_j\mu_i) \ . \tag{2.19}$$

The S matrix entries $S^{(N;n)}_{\lambda\mu}$ are most effectively expressed as an $N\times N$ determinant:

$$S^{(N;n)}_{\lambda\mu} = \beta \exp[2\pi \mathrm{i}\, t(\lambda)\, t(\mu)/(Nn)] \det(\exp[-2\pi \mathrm{i}\, \lambda[i]\, \mu[j]/n])_{1\le i,j\le N} \ , \tag{2.20}$$

where $\lambda[i] = \sum_{i\le \ell<N} (\lambda_\ell + 1)$ and *N-ality* $t(\lambda) := \sum_{j=1}^{N-1} j\lambda_j$. β is an irrelevant constant.

For any affine algebra, charge-conjugation C and simple-currents J correspond to symmetries of the associated Dynkin diagram. For SU(N), they act on Φ^n_N as follows:

$$\begin{aligned} C(\lambda_1, \dots, \lambda_{N-1}) &= (\lambda_{N-1}, \dots, \lambda_1) \ , && (2.21)\\ J(\lambda_1, \dots, \lambda_{N-1}) &= (\lambda_0, \lambda_1, \dots, \lambda_{N-2}) \ , && (2.22)\end{aligned}$$

where $\lambda_0 = n - \lambda_1 - \cdots - \lambda_{N-1}$. C is order 2 and J is order N. They obey (2.6),(2.7) as well as

$$\begin{aligned} T^{(N;n)}_{J^a\lambda, J^a\mu} &= \exp[\pi \mathrm{i}\, (a(N-a)n - 2at(\lambda))/N]\, T^{(N;n)}_{\lambda\mu} \ , && (2.23)\\ S^{(N;n)}_{J^a\lambda, J^b\mu} &= \exp[2\pi \mathrm{i}\, (bt(\lambda-\rho) + at(\mu-\rho) + nab)/N]\, S^{(N;n)}_{\lambda\mu} \ . && (2.24)\end{aligned}$$

The quantities Q_J and h_{J0} in Section 2.1 are thus $Q_J(\lambda) = t(\lambda-\rho)/N$ and $h_{J\rho} = (N-1)(n-N)/(2N)$. Note that $t(\rho) \equiv_N 0$ resp. $N/2$ for N odd resp. even, where throughout this paper '$x \equiv_m y$' abbreviates '$x \equiv y \pmod m$'. Also,

$$t(J^a\lambda - \rho) \equiv_N na + t(\lambda - \rho) \ . \tag{2.25}$$

Now specialise to $N=3$, the case of most interest to us. The T matrix for $\mathrm{su}_{3;n}$ is

$$T^{(n)}_{\lambda\mu} := T^{(3;n)}_{\lambda\mu} = \exp[2\pi \mathrm{i} \frac{\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 - n}{3n}]\, \delta_{\lambda\mu} \ . \tag{2.26}$$

The denominator identity of the Lie algebra A_2 implies, for $1 \le a < n/2$, the formula

$$S^{(n)}_{\lambda(a,a)} = S^{(n)}_{(a,a)\lambda} = \frac{8}{\sqrt{3}n} \sin\left(\frac{\pi a \lambda_1}{n}\right) \sin\left(\frac{\pi a \lambda_2}{n}\right) \sin\left(\frac{\pi a\,(\lambda_1+\lambda_2)}{n}\right) \ , \tag{2.27}$$

giving a convenient expression for the quantum-dimensions $\mathcal{D}^{(n)}\lambda := S^{(n)}_{\lambda\rho}/S^{(n)}_{\rho\rho}$.

Lemma 2.2 *(a) For all $\lambda, \mu \in \Phi^n_3$, $S^{(n)}_{(2,1)\lambda}/S^{(n)}_{\rho\lambda} = S^{(n)}_{(2,1)\mu}/S^{(n)}_{\rho\mu}$ iff $\lambda = \mu$.*

(b) Suppose $n \neq 4$. For any $\lambda \in \Phi^n_3$, $\lambda \notin \langle C, J\rangle(2,1) \cup \langle J\rangle\rho$,

$$\mathcal{D}^{(n)}\lambda > \mathcal{D}^{(n)}(2,1) > 1$$

(when $n = 4$, $\Phi^n_3 = \langle J\rangle\rho$).

Part (a) is a special case of Proposition 3 of [16] together with $S^{(n)}_{(1,2)\nu}/S^{(n)}_{\rho\nu} = \left(S^{(n)}_{(2,1)\nu}/S^{(n)}_{\rho\nu}\right)^*$. Part (b) is a special case of Proposition 1 of [16]. (See also the proof of Lemma 5.1 below.)

The Galois parity ϵ_ℓ in (2.9) satisfies

$$\epsilon^{(n)}_\ell(\lambda)\,\epsilon^{(n)}_\ell(\rho) = \begin{cases} +1 & \text{if } \{\ell\lambda_1\}_n + \{\ell\lambda_2\}_n < n \\ -1 & \text{if } \{\ell\lambda_1\}_n + \{\ell\lambda_2\}_n > n \end{cases} \tag{2.28}$$

for any ℓ coprime to $3n$, where $\{x\}_m$ is uniquely defined by $0 \le \{x\}_m < m$ and $x \equiv_m \{x\}_m$.

Lemma 2.3 *(a) [15] Suppose $\lambda \in \Phi^n_3$ satisfies $T^{(n)\ 3}_{\lambda\lambda} = T^{(n)\ 3}_{\rho\rho}$ and, for all ℓ coprime to $3n$, $\epsilon^{(n)}_\ell(\lambda) = \epsilon^{(n)}_\ell(\rho)$. Then:*

(i) for $n \equiv_4 1, 2, 3$, $n \neq 18 : \lambda \in \langle J\rangle\rho$;

(ii) for $n \equiv_4 0$, $n \neq 12, 24, 60 : \lambda \in \langle J\rangle\rho \cup \langle J\rangle\rho''$ where $\rho'' = (\frac{n-2}{2}, \frac{n-2}{2})$;

(iii) $n = 12 : \lambda \in \langle J\rangle\rho \cup \langle J\rangle(3,3) \cup \langle J\rangle(5,5)$;

$n = 18 : \lambda \in \langle J\rangle\rho \cup \langle C, J\rangle(4,1)$;

$n = 24 : \lambda \in \langle J\rangle\rho \cup \langle J\rangle(5,5) \cup \langle J\rangle(7,7) \cup \langle J\rangle(11,11)$;

$n = 60 : \lambda \in \langle J\rangle\rho \cup \langle J\rangle(11,11) \cup \langle J\rangle(19,19) \cup \langle J\rangle(29,29)$.

(b) [24] Suppose n is coprime to 6. Then $\epsilon^{(n)}_\ell(\lambda) = \epsilon^{(n)}_\ell(\kappa)$ for all ℓ coprime to $3n$, iff $\kappa \in \langle C, J\rangle\lambda$.

The modular invariants for SU(3) were classified in [13, 15], and are building blocks for those of both $\mathrm{SU}(3)\times\mathrm{SU}(3)$ and W_3. The *generic* $\mathrm{su}_{3;n}$ modular invariants, existing at any height $n\geq 4$, consist of $\mathbb{A}_n := I$; charge-conjugation $\mathbb{A}_n^* := C$; the simple-current modular invariant

$$(\mathbb{D}_n)_{\lambda,\mu} = \delta_{\mu,J^{n\,t(\lambda)}\lambda} \qquad \text{for } 3\nmid n\ , \tag{2.29}$$

$$= \begin{cases} \delta_{\mu\lambda}+\delta_{\mu,J\lambda}+\delta_{\mu,J^2\lambda} & \text{if } 3\mid t(\lambda) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 3\mid n\ , \tag{2.30}$$

given by $\mathbb{D}_n := M[J]$ in (2.10); and the matrix product $\mathbb{D}_n^* := C\mathbb{D}_n$. The remaining modular invariants, the *exceptionals*, in character notation (2.1) are

$$\begin{aligned}
\mathbb{E}_8 &= |\chi_\rho+\chi_{(3,3)}|^2+|\chi_{(1,3)}+\chi_{(4,3)}|^2+|\chi_{(3,1)}+\chi_{(3,4)}|^2 \\
&\quad +|\chi_{(3,2)}+\chi_{(1,6)}|^2+|\chi_{(4,1)}+\chi_{(1,4)}|^2+|\chi_{(2,3)}+\chi_{(6,1)}|^2\,, &(2.31)\\
\mathbb{E}_{12} &= |\chi_\rho+\chi_{(1,10)}+\chi_{(10,1)}+\chi_{(5,5)}+\chi_{(5,2)}+\chi_{(2,5)}|^2+2|\chi_{(3,3)}+\chi_{(3,6)}+\chi_{(6,3)}|^2\,, &(2.32)\\
\mathbb{E}'_{12} &= |\chi_\rho+\chi_{(10,1)}+\chi_{(1,10)}|^2+|\chi_{(3,3)}+\chi_{(3,6)}+\chi_{(6,3)}|^2+|\chi_{(1,4)}+\chi_{(7,1)}+\chi_{(4,7)}|^2 \\
&\quad +|\chi_{(4,1)}+\chi_{(1,7)}+\chi_{(7,4)}|^2+|\chi_{(5,5)}+\chi_{(5,2)}+\chi_{(2,5)}|^2+2|\chi_{(4,4)}|^2 \\
&\quad +(\chi_{(2,2)}+\chi_{(2,8)}+\chi_{(8,2)})\chi^*_{(4,4)}+\chi_{(4,4)}(\chi^*_{(2,2)}+\chi^*_{(2,8)}+\chi^*_{(8,2)})\,, &(2.33)\\
\mathbb{E}_{24} &= |\chi_\rho+\chi_{(5,5)}+\chi_{(7,7)}+\chi_{(11,11)}+\chi_{(22,1)}+\chi_{(1,11)} \\
&\qquad +\chi_{(14,5)}+\chi_{(5,14)}+\chi_{(11,2)}+\chi_{(2,11)}+\chi_{(10,7)}+\chi_{(7,10)}|^2 \\
&\quad +|\chi_{(1,7)}+\chi_{(7,1)}+\chi_{(1,16)}+\chi_{(16,1)}+\chi_{(7,16)}+\chi_{(16,7)} \\
&\qquad +\chi_{(5,8)}+\chi_{(8,5)}+\chi_{(5,11)}+\chi_{(11,5)}+\chi_{(8,11)}+\chi_{(11,8)}|^2\,, &(2.34)
\end{aligned}$$

at heights $n=8,12,12,24$ respectively, as well as the matrix products $\mathbb{E}_8^* := C\mathbb{E}_8$ and $\mathbb{E}'^*_{12} := C\mathbb{E}'_{12}$. Some curiousities about the SU(3) modular invariants are described in [1].

3 The modular invariants of $W_N(p,q)$

This section introduces the $W_N(p,q)$ classification problem and reduces its solution to that of $\mathrm{SU}(N)\times\mathrm{SU}(N)$ at height (p,q) (see Theorem 3.1, one of the main results of this paper). This is not at all an easy observation and to our knowledge nothing like this has appeared in the literature before. Section 3.3 uses this correspondence to rewrite the classification proof for Virasoro minimal models. The $\mathrm{SU}(3)\times\mathrm{SU}(3)$ modular invariant classification is given by Theorem 3.2 (though its proof is deferred to Sections 4 and 5). The complete list of $W_3(p,q)$ modular invariants (the other main result of this paper) is Theorem 3.3. Previously, only the *unitary* $W_3(p,q)$ modular invariants (i.e. the special case $q=p+1$) were classified [18].

3.1 The $W_N(p,q)$ modular data

Choose any integers $N\geq 2$ and $p,q>N$. We require p,q to be coprime. The W_N minimal model modular data is related to that of WZW models on $\mathrm{SU}(N)\times\mathrm{SU}(N)$. We abbreviate

'SU(N) × SU(N) at height (p,q)' by '$\mathrm{su}^2_{N;p,q}$'. A highest weight for $\mathrm{su}^2_{N;p,q}$ is a pair $\lambda\mu := (\lambda,\mu)$ in $\Phi^{p,q}_N := \Phi^p_N \times \Phi^q_N$. There are N^2 simple-currents for $\mathrm{su}^2_{N;p,q}$, namely J^iK^j in obvious notation. The $\mathrm{su}^2_{N;p,q}$ modular data is the tensor of that of $\mathrm{su}_{N;p}$ and $\mathrm{su}_{N;q}$:

$$S^{(N;p,q)}_{\lambda\mu,\kappa\nu} = S^{(N;p)}_{\lambda\kappa} S^{(N;q)}_{\mu\nu}\ , \qquad T^{(N;p,q)}_{\lambda\mu,\lambda\mu} = T^{(N;p)}_{\lambda\lambda} T^{(N;q)}_{\mu\mu}\ . \tag{3.1}$$

For Theorem 3.1 below, call an integer r *pq-admissible* if both

$$\ell' := rp - q \quad \text{and} \quad \ell'' := r^2 p + q \text{ are coprime to } 2N\,. \tag{3.2}$$

For example, when $N = 2$ or 3 we can (and will) take $r = 0$. For any N, p, q, there are many pq-admissible r: e.g. for each prime P dividing $2N$, put $r_P = 1$ if $P \mid pq$ and $r_P = P$ otherwise, then $r = \prod_P r_P$ works. Fix any pq-admissible r. Then (2.25) and $\gcd(\ell', N) = 1$ say each JK-orbit $\langle JK\rangle\lambda'\mu' = \{(J^i\lambda, K^i\mu) : 0 \le i < N\}$ has exactly one element $\lambda\mu$ in

$$\Phi^{p,q}_{WN} := \{\lambda\mu \in \Phi^{p,q}_N \ :\ t(\mu) \equiv_N r\, t(\lambda)\}\,. \tag{3.3}$$

Similarly, $\gcd(\ell'', N) = 1$ says each orbit $\langle J^{-r}K\rangle\lambda'\mu'$ has exactly one element in $\Phi^{p,q}_{WN}$.

The W_N minimal models are parametrised by coprime integers $p, q > N$. Their central charge is $c = (N-1)[1 - N(N+1)(p-q)^2/pq]$. $W_N(p,q)$ is unitary iff $|p-q| = 1$. A $W_N(p,q)$ primary is a JK-orbit $[\lambda\mu] := \langle JK\rangle\lambda\mu$. We will sometimes identify the $W_N(p,q)$ primaries with $\Phi^{p,q}_{WN}$. The vacuum is $[\rho\rho]$. The $W_N(p,q)$ modular data is

$$\begin{aligned} S_{[\lambda\mu][\kappa\nu]} &= \alpha' \exp[-2\pi\mathrm{i}\,\frac{t(\lambda)t(\nu) + t(\mu)t(\kappa)}{N}]\, S^{(N;p/q)}_{\lambda\kappa} S^{(N;q/p)}_{\mu\nu}\ , && (3.4)\\ T_{[\lambda\mu][\lambda\mu]} &= \beta' \exp[\pi\mathrm{i}\,\left((q\lambda - p\mu)^2\right)/(pq)]\,, && (3.5) \end{aligned}$$

where α' and β' are independent of $[\lambda\mu], [\kappa\nu]$, and $S^{(N;p/q)}$ is the matrix (2.20) for $\mathrm{su}_{N;n}$ formally evaluated at the fractional height $n = p/q$. The N simple-currents of $W_N(p,q)$ are generated by $J1$ with $Q_{J1}([\lambda\mu]) = (qt(\lambda) - pt(\mu))/N + (N-1)/2$ and $h_{[J\rho,\rho]} = (N-1)[pq + N\,(p+q)]/(2N)$.

3.2 The Galois shuffle for W_N

For most p, q, the $W_N(p,q)$ modular data is non-unitary: $S_{[\lambda\mu][\rho\rho]}$ can be negative. Normally this would be bad news, as basic tools needed in modular invariant classifications (e.g. Lemma 2.1) break down for non-unitary modular data. However for any $W_N(p,q)$ there is *unitary* modular data $\widehat{S}, \widehat{T}$ with an equivalent list of modular invariants, obtained from S, T by the *Galois shuffle* of [17]. The subtle argument is given in detail in Section 6 of [17] but only sketched below. Here we make the crucial observation that $\widehat{S}, \widehat{T}$ can be arranged to be rescaled submatrices of $\mathrm{su}^2_{N;p,q}$ modular data, permitting the association of $\mathrm{su}^2_{N;p,q}$ modular invariants with $W_N(p,q)$ ones:

Theorem 3.1 *Fix any pq-admissible r. Let M be any modular invariant for $W_N(p,q)$. Let $\widetilde{M}$ be the matrix indexed by $\Phi_N^{p,q}$ with entries*

$$\widetilde{M}_{J^{ar}\lambda K^{-a}\mu,J^{br}\kappa K^{-b}\nu}=\delta_{ab}M_{[\lambda\mu][\kappa\nu]}\ , \tag{3.6}$$

for any $\lambda\mu,\kappa\nu\in\Phi_{WN}^{p,q}$ and any $0\le a<N$. Then $\widetilde{M}$ is an $\mathrm{su}^2_{N;p,q}$ modular invariant. Conversely, an $\mathrm{su}^2_{N;p,q}$ modular invariant $\widetilde{M}$ is associated in this way to a (necessarily unique) $W_N(p,q)$ modular invariant M, iff $\widetilde{M}_{J^{-r}\rho K\rho,J^{-r}\rho K\rho}=1$.

Proof of Theorem. [17] proved that for any ℓ coprime to $2N$ satisfying both $\ell q\equiv_p 1$ and $\ell p\equiv_q 1$, $[J_o\sigma_\ell\rho,\sigma_\ell\rho]$ is the $W_N(p,q)$ primary o with minimal conformal weight, where σ_ℓ is the Galois permutation in (2.9), and $J_o=id.$ for N odd and $J_o=id.$ or $J^{N/2}$ for N even. The desired unitary modular data is

$$\widehat{S}_{[\lambda\mu][\kappa\nu]}:=\epsilon\,\sigma_\ell(S_{[J_o\lambda,\mu][J_o\kappa,\nu]})\ ,\qquad \widehat{T}_{[\lambda\mu][\kappa\nu]}:=\epsilon\,(T_{[J_o\lambda,\mu][J_o\kappa,\nu]})^\ell\ , \tag{3.7}$$

where $\epsilon\in\{\pm1\}$ is an irrelevant constant. Moreover, the identity $M_{[J_o\rho,\rho][J_o\rho,\rho]}=1$ (also proved in Section 6 of [17]) together with Consequence 2(viii) there ensures the bijection $M\leftrightarrow\widehat{M}$ between $W_N(p,q)$ and $\widehat{S},\widehat{T}$ modular invariants, where $\widehat{M}_{[\lambda\mu][\kappa\nu]}=M_{[\lambda\mu][\kappa\nu]}$. Lemma 2.1(a) now says that

$$\widehat{M}_{[J_o\lambda,\mu][J_o\kappa,\nu]}=\widehat{M}_{[\lambda\mu][\kappa\mu]}\ . \tag{3.8}$$

To go further, fix $\ell\equiv_{2N}\ell''\ell'^{-2}$. It obeys $\ell q=1+Ap$ and $\ell p=1+Bq$ where

$$A\equiv_{2N}\ell'^{-2}r\,(rq-rp+2pq)\text{ and }B\equiv_{2N}\ell'^{-2}\,(2rp+p-q)\,. \tag{3.9}$$

Incidentally, for N even, $J_o\ne id.$ here iff r is even. All pq-admissible r are even iff pq is odd.

We want to show that for any $\lambda\mu,\kappa\nu\in\Phi_{WN}^{p,q}$,

$$\widehat{S}_{[J_o\lambda,\mu][J_o\kappa,\nu]} = \alpha''\,S^{(N;p,q)}_{\lambda\mu,\kappa\nu}\ , \tag{3.10}$$

$$\widehat{T}_{[J_o\lambda,\mu][J_o\lambda,\mu]} = \beta''\,T^{(N;p,q)}_{\lambda\mu,\lambda\mu}\ , \tag{3.11}$$

where α'',β'' are independent of $\lambda\mu,\kappa\nu$. To see (3.11), first note from (2.19) that for any $\lambda,\mu\in\Phi_N^n$,

$$N\lambda^2 \equiv_{2N} (N-t(\lambda))\,t(\lambda)\ , \tag{3.12}$$

$$N\lambda\cdot\mu \equiv_N -t(\lambda)\,t(\mu)\ . \tag{3.13}$$

Take any $\lambda\mu\in\Phi_{WN}^{p,q}$, and write $s:=t(\lambda)$ and $t(\mu)=:rs+s'N$. Then

$$\begin{aligned}\widehat{T}_{[J_o\lambda\mu][J_o\lambda\mu]} &= \beta'''T^{(N;p,q)}_{\lambda\mu,\lambda\mu}\exp[\pi\mathrm{i}\,(A(N-s)s+2\ell s(rs+s'N)+B(N-rs-s'N)(rs+s'N))/N]\\ &= \beta''T^{(N;p,q)}_{\lambda\mu,\lambda\mu}(-1)^{A+Br}\exp[\pi\mathrm{i}\,(2\ell r-A-Br^2)s^2/N]\ .\end{aligned} \tag{3.14}$$

But both $2\ell r - A - Br^2 \equiv_{2N} 0$ and $A + Br \equiv_2 0$ follow automatically from (3.9) and (3.2), giving (3.11). To see (3.10), use (2.20) and (3.3) to write

$$\begin{aligned}\widehat{S}_{[J_o\lambda,\mu][J_o\lambda,\mu]} &= \alpha''' \exp[2\pi\mathrm{i}\frac{-2\ell r\, t(\lambda)\, t(\kappa)}{N} + \frac{\ell q\, t(\lambda)\, t(\kappa)}{Np} + \frac{\ell p\, t(\mu)\, t(\nu)}{Nq}]\, \sigma_\ell(S'_{\lambda\kappa}{}^{(N;p/q)} S'_{\mu\nu}{}^{(N;q/p)}) \\ &= \alpha'' \exp[2\pi\mathrm{i}\, t(\lambda)\, t(\kappa)\,(-2r\ell + A + Br^2)/N]\, S^{(N;p,q)}_{\lambda\mu,\kappa\nu}\ , \qquad (3.15)\end{aligned}$$

where S' denotes the determinant in (2.20). Of course $-2r\ell + A + Br^2 \equiv_N 0$ so we're done.

The proof that $\widetilde{M}$ defined above commutes with $S^{(N;p,q)}$ and $T^{(N;p,q)}$ is now an easy application of (2.24),(2.23),(3.3) and (3.8). That the condition $\widetilde{M}_{J^{-r}\rho K\rho, J^{-r}\rho K\rho} = 1$ ensures a corresponding M exists, follows from Lemma 2.1(a). QED

Corollary 3.1 *The modular invariants for $W_N(p, N+1)$ are in natural one-to-one bijection with those of $SU(N)$ at level $p - N$.*

Indeed, $\Phi^{N+1}_N = \langle J\rangle\rho$. Since $q = N + 1$ is coprime to N, take $r = 0$ in Theorem 3.1. Define $M'_{\lambda\kappa} = M_{[\lambda\rho][\kappa\rho]}$. Then (3.6) says $\widetilde{M}_{\lambda K^a\rho,\kappa K^b\rho} = \delta_{ab} M'_{\lambda\kappa}$, so $\widetilde{M}$ is an $\mathrm{su}^2_{N;p,q}$ modular invariant iff M' is an $\mathrm{su}_{N;p}$ one.

3.3 A warm-up exercise: the Virasoro minimal models

Restrict now to $N = 2$. The $\mathrm{su}_{2;n}$ modular data is given by

$$S_{ab} = \sqrt{2/n}\, \sin(\pi\, a\, b/n)\ , \qquad T_{aa} = \exp\left[\pi\mathrm{i}\, a^2/2n - \pi\mathrm{i}/4\right]\ . \qquad (3.16)$$

Charge-conjugation C is trivial and the vacuum is 1. The only nontrivial simple-current is $Ja = n - a$ with $Q_J(a) = (a-1)/2$ and $h_{J1} = (n-2)/4$. Recall the quantity $\{x\}_m$ of Section 2.2. For any ℓ coprime to $2n$, the parity $\epsilon_\ell(a)$ in (2.9) depends on an irrelevant contribution from $\mathrm{i}\sqrt{2/n}$, as well as the sign $+1$ or -1, respectively, depending on whether or not $\{\ell a\}_{2n} < n$.

The $\mathrm{su}_{2;n}$ modular invariants are: the identity I for all $n \geq 3$; the simple-current invariant

$$\begin{aligned} M[J]_n &= \sum_{1\leq a\leq n-1} \chi_a\, \chi^*_{J^{a-1}a} && \text{whenever } 4 \mid n\ , && (3.17)\\ &= |\chi_1 + \chi_{J1}|^2 + |\chi_3 + \chi_{J3}|^2 + \cdots + 2|\chi_{\frac{n}{2}}|^2 && \text{whenever } n \equiv_4 2\ ; && (3.18)\end{aligned}$$

as well as the exceptionals

$$\begin{aligned} \mathbb{E}^{A1}_{12} &= |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 && \text{for } n = 12\ , && (3.19)\\ \mathbb{E}^{A1}_{18} &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 && && \\ &\quad + \chi_9\,(\chi_3 + \chi_{15})^* + (\chi_3 + \chi_{15})\,\chi_9^* + |\chi_9|^2 && \text{for } n = 18\ , && (3.20)\\ \mathbb{E}^{A1}_{30} &= |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 && \text{for } n = 30\ . && (3.21)\end{aligned}$$

Cappelli-Itzykson-Zuber [8] (see also [9, 21, 23]) obtained the $W_2(p,q)$ classification, together with that of $\mathrm{su}_{2;n}$, by manifestly constructing a basis for the commutant, i.e. the space of all matrices commuting with S and T. Then they imposed (2.3) and (2.4). In this way without knowing the Galois shuffle (Theorem 3.1) they could still see the crucial fact that $M_{oo}=1$ and a correspondence between $W_2(p,q)$ and $\mathrm{su}^2_{2;p,q}$. Even so, their proof takes several pages and involves nontrivial number theory (e.g. that there is a prime between any m and $2m$). More important, $N=2$ behaves far simpler than $N>2$: the generalisation of their approach to W_3 (or even SU(3)) has still not been found even after years of effort.

A significantly simpler proof of the $\mathrm{su}_{2;n}$ modular invariant classification is provided in [19], using the ideas of Section 2.1. Likewise, Theorem 3.1 permits a much faster proof for $W_2(p,q)$ — indeed, much more falls out. In particular, Theorem 7 of [14] gives the $\mathrm{SU}(2)\times\cdots\times\mathrm{SU}(2)$ modular invariant classification for any height $(n_1,\ldots,n_s)$, provided only that $\gcd(n_i,n_j)\leq 3$ for all $i\neq j$. Specialising to $\mathrm{su}^2_{2;p,q}$ with p,q coprime (and say q odd) yields the answer: $I_p\otimes I_q$ for all p,q; $M[J]_p\otimes I_q$ for all even p; $\mathbb{E}^{A1}_p\otimes I_q$ for $p=12,18,30$; and finally

$$
\begin{aligned}
(M[JK]_{p,q})_{ab,cd} &= \delta_{c,J^{a+b}a}\delta_{d,K^{a+b}b} && \text{whenever } p\equiv_4 q\ , && (3.22)\\
&= \begin{cases}\delta_{ac}\delta_{bd}+\delta_{c,Ja}\delta_{d,Kb} & \text{if } a\equiv_2 b\\ 0 & \text{otherwise}\end{cases} && \text{whenever } p\equiv_4 -q\ . && (3.23)
\end{aligned}
$$

This classification can also be recovered quickly by following the approach of Sections 4 and 5 (see also [19]). For instance, the analogue of Lemma 2.3(a) is: If $\epsilon_\ell(a)=\epsilon_\ell(1)$ for all ℓ coprime to $2n$, then $a\in\{1,n-1\}$ unless: $n=6$ and $a\in\{1,3,5\}$; $n=10$ and $a\in\{1,3,7,9\}$; $n=12$ and $a\in\{1,5,7,11\}$; $n=30$ and $a\in\{1,11,19,29\}$. This is proved in a couple paragraphs in [19].

Choosing $r=0$ in Theorem 3.1, we are interested in all $\mathrm{su}^2_{2;p,q}$ modular invariants with $M_{(1,q-1),(1,q-1)}=1$. Clearly all M in factorised form $M'\otimes M''$, i.e. $M'\otimes I_q$, survive, but $M[JK]_{p,q}$ fails. This recovers the well-known result that the Virasoro minimal model modular invariants correspond to pairs M',M'' of SU(2) modular invariants.

3.4 The $\mathrm{SU}(3)\times\mathrm{SU}(3)$ and W_3 modular invariants

Theorem 3.1 says that the $W_3(p,q)$ modular invariants, our main interest, are determined once the $\mathrm{su}^2_{3;p,q}$ ones are. Because of (3.1), the latter include the tensor products $M'\otimes M''$ where M',M'' are modular invariants for $\mathrm{su}_{3;p}$ resp. $\mathrm{su}_{3;q}$. But tensor products won't exhaust all of them: e.g. the simple-current modular invariants

$$
\begin{aligned}
M[JK^{\pm1}]_{\lambda\mu,\kappa\nu} &= \delta_{\kappa,J^{(p+q)(t(\lambda)\pm t(\mu))}\lambda}\delta_{\nu,K^{(p+q)(t(\mu)\pm t(\lambda))}\mu} && \text{for } 3\nmid p+q\ , && (3.24)\\
&= \begin{cases}\sum_{0\leq i\leq 2}\delta_{\kappa,J^i\lambda}\delta_{\nu,K^{\pm i}\mu} & \text{if } t(\lambda)\equiv_3 \mp t(\mu)\\ 0 & \text{otherwise}\end{cases} && \text{for } 3\mid p+q\ , && (3.25)
\end{aligned}
$$

aren't of factorised form for either sign. Indeed, the modular invariant classification for $\mathrm{SU}(3)\times\mathrm{SU}(3)$ at arbitrary height (p,q) would be difficult to obtain (and probably not terribly interesting).

But much easier is when p, q are coprime (we can also insist without loss of generality that $3 \nmid q$ and $p \neq 8$). In this case the $\mathrm{su}^2_{3;p,q}$ classification contains nothing unexpected:

Theorem 3.2 *Let $p, q > 3$ be coprime, $3 \nmid q$, $p \neq 8$. The $\mathrm{su}^2_{3;p,q}$ modular invariants are:*

(*i*) *the tensor products $M' \otimes M''$ for any $su_{3;p}$, $su_{3;q}$ modular invariants M' resp. M'';*

(*ii*) *the products $(C^i \otimes C^j \mathbb{D}^l_q)\, M[JK^{\pm 1}]$ for any $i, j, l \in \{0, 1\}$ (C is SU(3) charge-conjugation);*

(*iii*) *when $p \equiv_3 1$ and $q = 8$, the exceptionals $(C^i \otimes C^j \mathbb{E}_8) M[JK^{\pm 1}]$ for any $i, j \in \{0, 1\}$;*

(*iv*) *when $p = 12$ and $q \equiv_3 -1$, the exceptionals $(C^i \otimes C^j)\mathbb{E}_{12,q}$.*

$\mathbb{E}_{12,q}$ is the matrix whose only nonzero entries are

$$(\mathbb{E}_{12,q})_{\lambda\mu,\kappa\mu} = (\mathbb{E}_{12,q})_{\lambda'\mu,\lambda K^a\mu} = (\mathbb{E}_{12,q})_{\lambda K^a\mu,\lambda'\mu} = (\mathbb{E}_{12,q})_{\lambda' K^a\mu,\kappa' K^b\mu} = 1 \tag{3.26}$$

for any $\lambda, \kappa \in [\rho] \cup [\rho'']$, $\lambda', \kappa' \in [(3,3)]$, $\mu \in \Phi^q_3$ with $3|t(\mu)$, and any $a, b \in \{\pm 1\}$. See (1.1) for another description of $\mathbb{E}_{12,5}$. This $\mathbb{E}_{12,q}$ is a modular invariant whenever $q \equiv_3 -1$ (q can be even). Proving modular invariance here reduces to verifying from (2.27) some simple identities obeyed by $S^{(12)}$, e.g. $S^{(12)}_{\rho(3,3)} = S^{(12)}_{(5,5)(3,3)} = S^{(12)}_{\rho\rho} + S^{(12)}_{\rho(5,5)} = -S^{(12)}_{(3,3)(3,3)}$.

Since $3 \nmid q$, take $r = 0$ in Theorem 3.1, so $\Phi^{p,q}_{W3}$ consists of all pairs $[\lambda\mu]$ with $3 \mid t(\mu)$. To obtain the W_3 classification, it suffices to check which $\mathrm{su}^2_{3;p,q}$ modular invariants $\widetilde{M}$ have $\widetilde{M}_{\rho K\rho,\rho K\rho} = 1$. For $\widetilde{M}$ in factorised form $M' \otimes M''$, this will happen iff $M'' = \mathbb{A}_q$, $\mathbb{D}^*_q$ or $\mathbb{E}_8$. If $3 \mid p + q$ then the whole $(\rho, K\rho)$-row of $M[JK^{\pm 1}]$ will vanish, whereas if $3 \nmid p + q$ then $M[JK^{\pm 1}]_{\rho K\rho, J^a\rho K^b\rho} = 1$ only for $a \neq 0$. Moreover, $(\mathbb{E}_{12,q})_{\rho K\rho, J^a\rho K^b\rho} = 0$ for all a, b. From those remarks it is easy to confirm that any $\widetilde{M}$ in Theorem 3.2(ii)-(iv) will fail the $\widetilde{M}_{\rho K\rho,\rho K\rho} = 1$ condition.

Thus any W_3 modular invariant M corresponds via (3.6) to an $\widetilde{M}$ in factorised form $M' \otimes M''$, completely analogously to Virasoro minimal models. Explicitly, such an M has entries

$$M_{[\lambda\mu][\kappa\nu]} = M'_{\lambda\kappa} M''_{\mu\nu} \tag{3.27}$$

where we restrict $[\lambda\mu], [\kappa\nu]$ to $\Phi^{p,q}_{W3}$, i.e. to $3 \mid t(\mu)$ and $3 \mid t(\nu)$.

We can regroup these using the *conjugations* $C^{i,j}_{W3}$, for $i, j, \in \{0, 1\}$, which act on $\Phi^{p,q}_{W3}$ by

$$C^{i,j}_{W3}[\lambda\mu] := [C^i\lambda, C^j\mu] \tag{3.28}$$

where C is again the SU(3) charge-conjugation (2.21). These $C^{i,j}_{W3}$ define $W_3(p, q)$ permutation invariants in the usual way, and thus multiplying by them sends a modular invariant to another.

Theorem 3.3 *Assume without loss of generality that $3 \nmid q$ and $p \neq 8$. The complete list of $W_3(p, q)$ modular invariants, up to left-multiplication by a conjugation $C^{i,j}_{W3}$, consists of the factorised modular invariants $\mathbb{A}_p \otimes \mathbb{A}_q$, $\mathbb{D}_p \otimes \mathbb{A}_q$, $\mathbb{E}_{12} \otimes \mathbb{A}_q$, $\mathbb{E}'_{12} \otimes \mathbb{A}_q$, $\mathbb{E}_{24} \otimes \mathbb{A}_q$, $\mathbb{A}_p \otimes \mathcal{E}_8$ and $\mathbb{D}_p \otimes \mathcal{E}_8$.*

As explained in Section 1, the exceptionals $(C^i \otimes C^j)\mathbb{E}_{12,q}$ are not the partition function of any RCFT. By Theorem 3.2, all other $\mathrm{su}^2_{3;p,q}$ modular invariants are either tensors of SU(3) ones, simple-current modular invariants $M[JK^{\pm 1}]$, or matrix products of these, so they will all be realised by subfactors in the sense of [4]. (This strongly suggests that they should also be realised in the framework of [28].) Among other things, this means they'll have a nim-rep. The corresponding $\widehat{M}$ (when it exists) for the modular data $\widehat{S}, \widehat{T}$ in (3.10),(3.11) will inherit this nim-rep. By Consequence 3(i) of [17], any of the W_3 modular invariants M have the same nim-reps as the corresponding $\widehat{M}$. For this reason we'd expect all W_3 modular invariants to give rise to an RCFT.

4 The SU(3)×SU(3) modular invariant classification

This section proves the $\mathrm{su}^2_{3;p,q}$ modular invariant classification for most p, q. Sections 2.2 and 3.1 fix our notation. For the remainder of this paper we restrict to coprime p, q; without loss of generality we assume $3 \nmid q$ and $p \neq 8$ (simplifying the bookkeeping). It would have been much faster to impose from the start the W_3 condition $M_{\rho K\rho,\rho K\rho} = 1$ but, as the exceptionals (3.26) indicate, the full $\mathrm{su}^2_{3;p,q}$ classification is itself of some interest.

4.1 The vacuum couplings

Let M be an $\mathrm{su}^2_{3;p,q}$ modular invariant, for p, q as above. When $\lambda\mu,\ \kappa\nu \in \Phi_3^{p,q}$ have $M_{\lambda\mu,\kappa\nu} \neq 0$, we say that $\lambda\mu$ and $\kappa\nu$ *M-couple*. The hardest step in modular invariant classifications is usually Step 1: to find which weights couple to $\rho\rho$. In the case of $\mathrm{su}^2_{3;p,q}$, this step follows from Lemma 2.3(a). Indeed, (3.1) implies that $\mathrm{su}^2_{3;p,q}$ Galois parities are products $\epsilon_\ell^{(p)}(\lambda)\,\epsilon_\ell^{(q)}(\mu)$ of SU(3) ones. By the Chinese Remainder Theorem, the parity rule (2.14) is equivalent to

$$M_{\lambda\mu,\kappa\nu} \neq 0 \Rightarrow \text{both } \epsilon_{\ell'}^{(p)}(\lambda) = \epsilon_{\ell'}^{(p)}(\kappa) \text{ and } \epsilon_{\ell''}^{(q)}(\mu) = \epsilon_{\ell''}^{(q)}(\nu)\ , \tag{4.1}$$

for all ℓ' coprime to $3p$ and all ℓ'' coprime to $3q$. The reason is that for any such ℓ', there will be an ℓ coprime to $3pq$ such that $\ell \equiv_p \ell'$ and $\ell \equiv_q 1$ (similarly for any such ℓ''). Moreover, (2.26) and $\gcd(p, q) = 1$ imply from (2.11) the selection rule

$$M_{\lambda\mu,\kappa\nu} \neq 0 \Rightarrow \text{both } (T^{(p)}_{\lambda\lambda})^3 = (T^{(p)}_{\kappa\kappa})^3 \text{ and } (T^{(q)}_{\mu\mu})^3 = (T^{(q)}_{\nu\nu})^3\ . \tag{4.2}$$

We are now ready for the main result of this subsection. Recall the notation $\langle J^aK^b\rangle$ for the subgroup of simple-currents and $\langle J^aK^b\rangle\lambda\mu$ for the corresponding orbit.

Lemma 4.1 *Let $p, q > 3$ be coprime, $p \neq 8$, and $3 \nmid q$. Define $\mathcal{R}_R(M) = \{\lambda\mu \in \Phi_3^{p,q} : M_{\rho\rho,\lambda\mu} \neq 0\}$ and $\mathcal{R}_L(M) = \{\lambda\mu \in \Phi_3^{p,q} : M_{\lambda\mu,\rho\rho} \neq 0\}$, and $\rho'' := ((p-2)/2, (p-2)/2)$, and let $\mathcal{J}_R(M)$ (resp. $\mathcal{J}_L(M)$) consist of the simple-currents J^aK^b in $\mathcal{R}_R$ (resp. $\mathcal{R}_L$). Then each $M_{\lambda\mu,\rho\rho}, M_{\rho\rho,\lambda\mu} \in \{0,1\}$, the possibilities for $\mathcal{J}_R = \mathcal{J}_R(M)$ and $\mathcal{J}_L = \mathcal{J}_L(M)$ are*

(i) $\mathcal{J}_R = \mathcal{J}_L = \{11\}$,

(ii) *when* $3 \mid p : \mathcal{J}_R = \mathcal{J}_L = \langle J1 \rangle$,

(iii) *when* $3 \mid p+q : \mathcal{J}_R, \mathcal{J}_L \in \{\langle JK \rangle, \langle JK^2 \rangle\}$,

and the possibilities for $\mathcal{R}_R = \mathcal{R}_R(M)$ *and* $\mathcal{R}_L = \mathcal{R}_L(M)$ *are*

(a) $\mathcal{R}_R = \mathcal{J}_R\rho\rho$, $\mathcal{R}_L = \mathcal{J}_L\rho\rho$,

(b) *when* $q = 8 : \mathcal{R}_R = \mathcal{J}_R\rho\rho \cup \mathcal{J}_R\rho\rho''$, $\mathcal{R}_L = \mathcal{J}_L\rho\rho \cup \mathcal{J}_L\rho\rho''$,

(c) *when* $p = 12 : \mathcal{R}_R = \mathcal{R}_L = \langle J1 \rangle\rho\rho \cup \langle J1 \rangle\rho''\rho$,

(d) *when* $p = 24 : \mathcal{R}_R = \mathcal{R}_L = \langle J1 \rangle\rho\rho \cup \langle J1 \rangle\rho'\rho \cup \langle J1 \rangle\rho''\rho \cup \langle J1 \rangle\rho'''\rho$, *for* $\rho' = (5,5)$, $\rho''' = (7,7)$.

Proof of Lemma. It suffices to look say at $\mathcal{R}_R$ and $\mathcal{J}_R$ (the equality $(MS)_{\rho\rho,\rho\rho} = (SM)_{\rho\rho,\rho\rho}$ then pegs $\mathcal{R}_L$ and $\mathcal{J}_L$ to $\mathcal{R}_R, \mathcal{J}_R$). If $J^aK^b \in \mathcal{J}_R$, then $a^2p + b^2q \equiv_3 0$ by (2.23). The only nontrivial solutions to this occur when $3 \mid p$ and $b = 0$, or $3 \mid p+q$ and $a = \pm b$ for either choice of sign. By Lemma 2.1(a), $\mathcal{J}_R$ forms a group, $\mathcal{J}_R\mathcal{R}_R = \mathcal{R}_R$, and for all $J^aK^b \in \mathcal{J}_R$,

$$M_{\rho\rho,J^a\lambda K^b\mu} = M_{\rho\rho,\lambda\mu} \tag{4.3}$$

(so $M_{\rho\rho,J^a\rho K^b\rho} = 1$). Among other things, this gives us (i)-(iii). It also tells us that when $\mathcal{R}_R \subseteq \langle J \rangle\rho \times \langle K \rangle\rho$, then $\mathcal{R}_R = \mathcal{J}_R\rho\rho$. Write $m = \|\mathcal{J}_R\|$, so $m = 1$ or 3.

Let $\lambda\mu \in \mathcal{R}_R$, and suppose $\lambda\mu$ is not a simple-current. Then Lemma 2.3(a) lists the candidates for p, q and λ, μ. We will use the condition $s_R(\kappa\nu) \geq 0$ (recall Lemma 2.1(b)) for specific $\kappa\nu$ to prove that either p or q must be one of $8, 12, 24$. All S-matrix entries we need are computed from (2.27), together with (2.24) and (2.6).

First, suppose $p = 18$. Then $4 \nmid q$ so $\mu = K^b\rho$ for some b. Then

$$0 \leq s_R((8,8)\rho) = mS^{(18)}_{\rho(8,8)}S^{(q)}_{\rho\rho} + 2m\,(M_{\rho\rho,(4,1)K\rho} + M_{\rho\rho,(4,1)K^2\rho})S^{(18)}_{(4,1)(8,8)}S^{(q)}_{\rho\rho}$$

(the factor of m resp. 2 arises from (4.3) resp. (2.12)). But $S^{(18)}_{(4,1)(8,8)}/S^{(18)}_{\rho(8,8)} \approx -1.22$, so positivity of the $S^{(n)}_{\rho\kappa}$ then forces all $C^iJ^a(4,1) \notin \mathcal{R}_R$, a contradiction.

Now consider $4 \nmid p$. Then $\lambda \in \langle J \rangle\rho$, $4 \mid q$, $\mu \in \langle K \rangle\rho''$, and $m' := \sum_{a,b} M_{\rho\rho,J^a\rho K^b\rho''} > 0$. When $q = 4$ $\rho'' = \rho$, so assume $q \geq 8$. By (4.3), m' is a multiple of m, hence $m' \geq m$. Then

$$\begin{aligned} 0 &\leq s_R(\rho, K^q(1,2)) = mS^{(p)}_{\rho\rho}S^{(q)}_{\rho(1,2)} + m'S^{(p)}_{\rho\rho}S^{(q)}_{\rho''(1,2)} \\ &= S^{(p)}_{\rho\rho}\frac{8}{\sqrt{3}q}\sin\left(\frac{2\pi}{q}\right)\left\{m\sin\left(\frac{\pi}{q}\right)\sin\left(\frac{3\pi}{q}\right) - m'\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{3\pi}{q}\right)\right\} . \end{aligned} \tag{4.4}$$

But the right side of (4.4) is manifestly negative for $q \geq 12$. When $q = 8$, (4.4) reduces to

$$m \sin(\pi/8)\sin(3\pi/8) \geq m' \sin(3\pi/8)\sin(\pi/8) \ .$$

This forces $m' = m$, i.e. $\mathcal{R}_R = \mathcal{J}_R(\rho\rho) \cup \mathcal{J}_R(\rho\rho'')$ and $M_{\rho\rho,\rho\rho''} = 1$.

Finally consider $4 \mid p$. Then q is odd, and the identical argument forces $p = 12, 24, 60$ and (using (2.26)) $\mu = \rho$. The argument given in Section 6 of [15] now holds without change (the $\mathrm{su}_{3;q}$ component ρ comes along for the ride). QED

4.2 The permutation invariants

Recall from Section 2.1 that the permutation invariants are the modular invariants satisfying $M_{\lambda\mu} = \delta_{\mu,\pi\lambda}$ for some permutation π. The permutation invariants for $\mathrm{SU}(N_1) \times \cdots \times \mathrm{SU}(N_s)$ at arbitrary heights were classified in [16]. From this we read off that every permutation invariant of $\mathrm{su}^2_{3;p,q}$ is a product $(C^i \otimes C^j)\pi_a$ for some $i, j \in \{0, 1\}$ and some π_a, where π_a is defined as follows. To any $a = (a_{11}, a_{21}, a_{12}, a_{22}) \in \mathbb{Z}_3^4$ obeying

$$a_{lm} + a_{ml} + pa_{l1}a_{m1} + qa_{l2}a_{m2} \equiv_3 0 \tag{4.5}$$

for all $l, m \in \{1, 2\}$, define

$$\pi_a(\lambda, \mu) = (J^{a_{11}t(\lambda)+a_{21}t(\mu)}\lambda, K^{a_{12}t(\lambda)+a_{22}t(\mu)}\mu) \ , \tag{4.6}$$

where $t(\lambda)$ is the triality defined in Section 2.2. The solutions to (4.5) are as follows. When $3 \mid p$, there are 6 solutions, namely $a_{11} = qa_{12}^2$ and $a_{21} = -a_{12}(1 + qa_{22})$ for any $a_{22} \not\equiv_3 -q$ and any a_{12}. These are all generated by $I \otimes M[K]$ and $M[JK^{\pm 1}]$ and are included in Theorem 3.2(i)-(ii). When $3 \mid p + q$, there are 4 solutions, namely any $a_{11} \not\equiv_3 -p$, $a_{22} \not\equiv_3 -q$, and $a_{12} \equiv_3 a_{21} \equiv_3 0$. These are generated by $M[J] \otimes I$ and $I \otimes M[K]$ and are included in Theorem 3.2(i). Finally, when $3 \mid p - q$, there are 8 solutions, namely any $a_{11}, a_{22} \in \{0, p\}$ and $a_{12} \equiv_3 a_{21} \equiv_3 0$, as well as any $a_{12}, a_{21} \in \{1, -1\}$ and $a_{11} \equiv_3 a_{22} \equiv_3 -p$. These are generated by $M[J] \otimes I, I \otimes M[K], M[JK^{\pm 1}]$ and again are included in Theorem 3.2(i)-(ii).

4.3 The simple-current extensions when $3 \nmid p$

The remainder of this section completes Step 2 of the $\mathrm{su}^2_{3;p,q}$ classification. As always, p, q are coprime and $3 \nmid q$. Let M be any modular invariant satisfying both $\mathcal{R}_L(M) = \mathcal{J}_L(M)$ and $\mathcal{R}_R(M) = \mathcal{J}_R(M)$ (these are defined in Lemma 4.1). Note that this is automatic if neither p nor q is one of 8, 12, or 24 (these exceptional heights are dealt with in Section 5). We may also assume that $\|\mathcal{J}_R\| = \|\mathcal{J}_L\| = 3$ (otherwise M is a permutation invariant). By Lemma 4.1, there are two possibilities: either $3 \mid p + q$ and $\mathcal{R}_L, \mathcal{R}_R \in \{\langle JK\rangle\rho\rho, \langle JK^2\rangle\rho\rho\}$ (handled in this subsection); or $3 \mid p$ and $\mathcal{R}_R = \mathcal{R}_L = \langle J1\rangle\rho\rho$ (handled in the next). The latter case is more difficult as it involves *fixed-points* (i.e. primaries fixed by nontrivial simple-currents in $\mathcal{J}_R$ or $\mathcal{J}_L$).

The following result, used in both this subsection and the next, is Lemma 3 of [15].

Lemma 4.2 *[15] Let M be any modular invariant for $\mathrm{su}^2_{3;p,q}$ with $\mathcal{R}_L = \mathcal{J}_L\rho\rho$ and $\mathcal{R}_R = \mathcal{J}_R\rho\rho$, and suppose that $M_{\lambda\mu,\kappa\nu} \neq 0$. Then*

$$M_{\lambda\mu,\kappa\nu} \leq \|\mathcal{J}_L\| / \sqrt{\|\mathcal{J}_L\lambda\mu\| \ \|\mathcal{J}_R\kappa\nu\|} \ .$$

If $\lambda\mu$ resp. $\kappa\nu$ are not fixed-points of $\mathcal{J}_L$ resp. $\mathcal{J}_R$, then $M_{\lambda\mu,\kappa\nu} = 1$ and, in addition, $M_{\lambda\mu,\alpha\beta} \neq 0$ iff $\alpha\beta \in \mathcal{J}_R(\kappa\nu)$ (similarly for $M_{\alpha\beta,\kappa\nu} \neq 0$).

Now consider $3 \mid p+q$ and $\mathcal{R}_R, \mathcal{R}_L \in \{\langle JK\rangle\rho\rho, \langle JK^2\rangle\rho\rho\}$. If $\mathcal{J}_R \neq \mathcal{J}_L$, hit M on the left by the permutation invariant $I \otimes \mathbb{D}_q$. If $\mathcal{J}_R = \mathcal{J}_L = \langle JK^2\rangle$, it is also convenient to multiply both left and right sides of M by $I \otimes C$. Hence, without loss of generality, we can assume $\mathcal{J}_R = \mathcal{J}_L = \langle JK\rangle =: \mathcal{J}$. By Lemma 2.1(b), the $\lambda\mu$-row of M will be identically zero iff $t(\lambda) \not\equiv_3 -t(\mu)$ iff the $\lambda\mu$-column of M is identically zero (recall triality t from Section 2.2). Let P_{00} denote the set of all pairs $\lambda\mu \in \Phi_3^{p,q}$ such that $t(\lambda) \equiv_3 t(\mu) \equiv_3 0$. Then any orbit $\mathcal{J}\lambda\mu$ with $t(\lambda) \equiv_3 -t(\mu)$ intersects P_{00} in exactly 1 point. Because there are no fixed-points here, Lemma 4.2 says that there is a permutation π of P_{00} which completely determines M, in the sense that, for any $\lambda\mu, \kappa\nu \in \Phi_3^{p,q}$, $M_{\lambda\mu,\kappa\nu} \neq 0$ iff $M_{\lambda\mu,\kappa\nu} = 1$ iff there is a $\lambda'\mu' \in P_{00}$ such that both $\lambda\mu \in \mathcal{J}\lambda'\mu'$ and $\kappa\nu \in \mathcal{J}\pi(\lambda'\mu')$. For $q = 4$, the second component of π is trivially the identity, so consider $q > 4$.

We study M through its π. The key fact is that, for all $\lambda\mu, \kappa\nu \in P_{00}$, $SM = MS$ becomes

$$S^{(p,q)}_{\lambda\mu,\kappa\nu} = S^{(p,q)}_{\lambda'\mu',\kappa'\nu'} \ , \tag{4.7}$$

where we write $\pi(\lambda\mu) = \lambda'\mu'$ and $\pi(\kappa\nu) = \kappa'\nu'$. Among other things, this tells us the identity

$$\mathcal{D}^{(p)}\lambda \ \mathcal{D}^{(q)}\mu = \mathcal{D}^{(p)}\lambda' \ \mathcal{D}^{(q)}\mu' \tag{4.8}$$

among quantum-dimensions $\mathcal{D}^{(n)}\kappa := S^{(n)}_{\kappa\rho}/S^{(n)}_{\rho\rho}$. Write $\pi(\rho, K^{-q}(2,1)) = \lambda'\mu'$. Suppose for contradiction that $\lambda' := (a,b) \neq \rho$. Then by Lemma 2.2(b), $\mu' = \rho$, and (2.11) yields

$$(a^2 + b^2 + ab - 3)/p \equiv_1 4/q \ , \tag{4.9}$$

an impossibility since p, q are coprime and $q > 4$. Therefore we must have $\lambda' = \rho$, so Lemma 2.2(b) together with (2.11) forces either $\mu' = K^{-q}(2,1)$ or $\mu' = K^q(1,2)$. Hence we may assume $\pi(\rho K^{-q}(2,1)) = \rho K^{-q}(2,1)$, multiplying on the left if necessary by $(I \otimes C)(I \otimes \mathbb{D}_q)$. Similarly, we may assume $\pi(J^{-p}(2,1)\rho) = J^{-p}(2,1)\rho$, if necessary multiplying on the left by $(C \otimes I)(\mathbb{D}_p \otimes I)$.

Now choose any $\lambda\mu \in P_{00}$ and write $\pi(\lambda\mu) = \lambda'\mu'$ as before. Comparing (4.7) for $\kappa\nu = \rho\rho$ and $\kappa\nu = \rho(2,1)$ gives

$$S^{(q)}_{\mu(2,1)}/S^{(q)}_{\mu\rho} = S^{(q)}_{\mu'(2,1)}/S^{(q)}_{\mu'\rho} \ . \tag{4.10}$$

Then Lemma 2.2(a) says $\mu = \mu'$. Using instead $\kappa\nu = (2,1)\rho$ likewise gives $\lambda = \lambda'$. Thus $\pi(\lambda\mu) = \lambda\mu$ and $M = M[JK]$. Undoing all of the left-multiplications by permutation invariants, we recover all of Theorem 3.2(ii) when $3 \mid p+q$.

4.4 The simple-current extensions when $3 \mid p$

Now consider $3 \mid p$ and $\mathcal{J}_L = \mathcal{J}_R = \langle J1 \rangle$. Here we have fixed-points, namely $\phi\mu$ for any $\mu \in \Phi_3^q$, where $\phi := (\frac{p}{3}, \frac{p}{3})$. By Lemma 2.1(b), for any $\lambda\mu \in \Phi_3^{p,q}$, the $\lambda\mu$-row will be nonzero iff $3 \mid t(\lambda)$ (similarly for the columns). Let $\mathcal{P}_0$ be the set of $\langle J1 \rangle$-orbits $[\lambda]\mu$, for all $\lambda\mu \in \Phi_3^{p,q}$ with $3 \mid t(\lambda)$.

Lemma 4.2 says that if a nonfixed-point $[\lambda]\mu \in \mathcal{P}_0$ does not couple to a fixed-point, then $M_{[\lambda]\mu,[\lambda']\mu'} = 1$ for a unique nonfixed-point orbit $[\lambda']\mu' \in \mathcal{P}_0$ (all other entries $M_{[\lambda]\mu,*} = 0$). Whenever nonfixed-points $[\lambda]\mu, [\kappa]\nu \in \mathcal{P}_0$ couple to nonfixed-points $[\lambda']\mu', [\kappa']\nu'$, then (4.7) holds.

First note that every $[\rho]\mu \in \mathcal{P}_0$ must couple to a nonfixed-point (otherwise (2.11) would give

$$3/p \equiv_1 (a'^2 + a'b' + b'^2 - a^2 - ab - b^2)/q \ ,$$

for $\mu = (a, b), \mu' = (a', b')$, contradicting $p, q > 3$ coprime). Say $M_{[\rho]\mu,[\lambda']\mu'} = 1$. Then for the choice $\mu = (2, 1)$, (4.7),(2.11) and Lemma 2.2(b) force $[\lambda'] = [\rho]$, by a similar argument to (4.9). Thus Lemma 2.2(b) requires $\mu' = K^a(2, 1)$ for some a, hitting M on the left if necessary by $I \otimes C$. Now (4.7) for $\lambda\mu = \kappa\nu = \rho(2, 1)$, together with (2.24), reads

$$S^{(q)}_{(2,1)(2,1)} = S^{(q)}_{K^a(2,1),K^a(2,1)} = \exp[2\pi \mathrm{i}\, (-a + qa^2)/3] S^{(q)}_{(2,1)(2,1)} \ .$$

Since $S^{(q)}_{(2,1)\lambda} \neq 0$ for any nonfixed-point λ (this is immediate from Lemma 2.2(a)), we must have $a = 0$ or q. Thus hitting M if necessary by $I \otimes \mathbb{D}_q$, we can assume in fact that $M_{[\rho](2,1),[\rho](2,1)} = 1$.

Now choose any $[\rho]\mu$ and write $M_{[\rho]\mu,[\lambda']\mu'} = 1$. Comparing (4.7) for $\kappa\nu = \rho\rho$ and $\kappa\nu = \rho(2, 1)$ gives (4.10) and hence $\mu = \mu'$; now (4.8) forces $[\lambda'] = [\rho]$. Thus we know $M_{[\rho]\mu,[\rho]\mu} = 1$ for all μ.

Lemma 4.3 *Let M be any $\mathrm{su}^2_{3;p,q}$ modular invariant satisfying $M_{\rho\mu,\kappa\nu} = M''_{\mu\nu} M_{\rho\rho,\kappa\rho}$ for all μ, κ, ν, where M'' is one of $\mathbb{A}_q, \mathbb{A}^*_q, \mathbb{D}_q$ or $\mathbb{D}^*_q$. Then $M = M' \otimes M''$ for some $su_{3;p}$ modular invariant M'.*

Proof of Lemma. First evaluate $MS = SM$ at $(\rho\tau, \lambda\mu)$ and commute $S^{(q)}$ through M'':

$$\sum_{\kappa,\nu} M_{\rho\rho,\kappa\rho} S^{(p)}_{\kappa\lambda} S^{(q)}_{\tau\nu} M''_{\nu\mu} = \sum_{\kappa,\nu} S^{(p)}_{\rho\kappa} S^{(q)}_{\tau\nu} M_{\kappa\nu,\lambda\mu} \ .$$

Now hit both sides with $S^{(q)*}_{\mu'\tau}$ and sum over τ:

$$M''_{\mu'\mu} \sum_{\kappa} M_{\rho\rho,\kappa\rho} S^{(p)}_{\kappa\lambda} = \sum_{\kappa} S^{(p)}_{\rho\kappa} M_{\kappa\mu',\lambda\mu} \ .$$

Hence $M_{\kappa\mu',\lambda\mu} = 0$ unless $M''_{\mu'\mu} \neq 0$. For those pairs μ, μ', define matrices $M'(\mu')$ by $M'(\mu')_{\lambda\kappa} = M_{\lambda\mu',\kappa\mu}/M''_{\mu',\mu}$ (for the possible M'' listed in Lemma 4.3, μ is determined by μ' up to simple-currents $\mathcal{J}_R(M'')$, so $M'(\mu')$ is indeed independent of μ, using Lemma 2.1(a)).

Evaluating $MS = SM$ at $(\lambda\mu, \kappa\rho)$ gives $M'(\mu)\, S^{(p)} = S^{(p)}\, M'(\rho)$, i.e. $M'(\mu) = S^{(p)}\, M'(\rho)\, S^{(p)*}$ is independent of μ and commutes with $S^{(p)}$. Likewise it commutes with $T^{(p)}$ and thus defines a modular invariant M' of $\mathrm{su}_{3;p}$. QED

Hence our M factorises into $M'\otimes I$ where M' satisfies $\mathcal{R}_L(M')=\mathcal{R}_R(M')=\langle J\rangle\rho$, i.e. $M'=\mathbb{D}_p$ or $\mathbb{E}'_{12}$, and we're done. Undoing the left-multiplications yields some of the M's in Theorem 3.2(i).

(Of course the Lemma also holds with the roles of p and q interchanged. Lemma 4.3 holds more generally whenever the modular data is factorisable: $S=S'\otimes S''$ and $T=T'\otimes T''$ — e.g. coprimedness is not needed.)

5 The exceptional modular invariants of SU(3)×SU(3)

This section handles Step 3: M is a modular invariant of $\mathrm{su}^2_{3;p,q}$ with $\mathcal{R}_L(M)\neq\mathcal{J}_L(M)$ and $\mathcal{R}_R(M)\neq\mathcal{J}_R(M)$. By Lemma 4.1, this can only happen when one of p or q is 8, 12, or 24. Recall that we require $p\neq 8$.

5.1 $q=8$ and $\|\mathcal{R}\|=2$

Lemma 4.1 says the only nonzero entries in the $\rho\rho$-row and column are $M_{\rho\rho,\rho\rho}=M_{\rho\rho,\rho\rho''}=M_{\rho\rho'',\rho\rho}=1$. Suppose $M_{\lambda\mu,\kappa\nu}\neq 0$. By Lemma 2.1(b), $\mu,\nu\in\langle K\rangle\rho\cup\langle K\rangle\rho''\cup\langle J,C\rangle(4,1)$. From (2.11) then, we obtain that $\mu\in\langle K\rangle\rho\cup\langle K\rangle\rho''$ iff $\nu\in\langle K\rangle\rho\cup\langle K\rangle\rho''$.

Evaluating $MS=SM$ at $(\lambda K^a\rho,\rho\rho)$ gives

$$\sum_{\lambda',b}M_{\lambda K^a\rho,\lambda'K^b\rho}S^{(p)}_{\lambda'\rho}S^{(8)}_{\rho\rho}+\sum_{\lambda'',c}M_{\lambda K^a\rho,\lambda''K^c\rho''}S^{(p)}_{\lambda''\rho}S^{(8)}_{\rho''\rho}=S^{(p)}_{\lambda\rho}S^{(8)}_{\rho\rho}+S^{(p)}_{\lambda\rho}S^{(8)}_{\rho\rho''}\ . \tag{5.1}$$

But $\mathcal{D}^{(8)}\rho''=3+2\sqrt{2}$, so $S^{(8)}_{\rho\rho}$ and $S^{(8)}_{\rho\rho''}$ are linearly independent over $\mathbb{Q}[e^{2\pi\mathrm{i}/3p}]$, where the $S^{(p)}$ entries lie. Therefore, equating coefficients, we obtain

$$\sum_{\lambda',b}M_{\lambda K^a\rho,\lambda'K^b\rho}S^{(p)}_{\lambda'\rho}=S^{(p)}_{\lambda\rho}=\sum_{\lambda'',c}M_{\lambda K^a\rho,\lambda''K^c\rho''}S^{(p)}_{\lambda''\rho}\ . \tag{5.2}$$

Then $M_{\lambda K^a\rho,\lambda'K^b\rho}\neq 0$ forces $\mathcal{D}^{(p)}\lambda\geq\mathcal{D}^{(p)}\lambda'$. But dually, we'd also get $\mathcal{D}^{(p)}\lambda'\geq\mathcal{D}^{(p)}\lambda$. Hence $\mathcal{D}^{(p)}\lambda=\mathcal{D}^{(p)}\lambda'$ (and likewise $\mathcal{D}^{(p)}\lambda=\mathcal{D}^{(p)}\lambda''$), and only one term (λ',b) and (λ'',c) can appear nontrivially in the sums (5.2).

The Galois automorphism σ_ℓ corresponding to $\ell'=1,\ell''=-1$ (recall Section 4.1) says $M_{\lambda\mu,\kappa\nu}=M_{\lambda C\mu,\kappa C\nu}$; this then means that when $a=0$, we must have $b=c=0$ (or the uniqueness of last paragraph would be violated). So we have learned that for each λ, there are unique λ',λ'' such that $\mathcal{D}^{(p)}\lambda=\mathcal{D}^{(p)}\lambda'=\mathcal{D}^{(p)}\lambda''$ and the only nonzero entries of the $\lambda\rho$-row of M are $M_{\lambda\rho,\lambda'\rho}=M_{\lambda\rho,\lambda''\rho''}=1$. But evaluating $MS=SM$ at $(\lambda\rho,\kappa(2,1))$ for any κ, implies $S^{(p)}_{\lambda'\kappa}=S^{(p)}_{\lambda''\kappa}$ $\forall\kappa$, hence $\lambda'=\lambda''$ for all λ.

Of course we can interchange the roles of rows and columns, and we find that the only nonzero entries of the $(2,1)\rho$-column are $M_{'(2,1)\rho,(2,1)\rho}=M_{'(2,1)\rho'',(2,1)\rho}=1$ for some $'(2,1)\in\Phi^n_3$ with $\mathcal{D}^{(p)}('(2,1))=\mathcal{D}^{(p)}(2,1)$. Lemma 2.2(b) tells us $'(2,1)=C^aJ^b(2,1)$ for some a,b. But (2.11)

forces $b=0$ or (when $3\nmid p$) $b=p$. Therefore hitting M on the left if necessary by the permutation invariants $C\otimes I$ and/or (for $3\nmid p$) $\mathbb{D}_p\otimes I$, we may assume in fact that $'(2,1)=(2,1)$.

Now evaluating $MS=SM$ at $(\lambda\rho,(2,1)\rho)$ gives $S^{(p)}_{\lambda'(2,1)}=S^{(p)}_{\lambda(2,1)}$, so Lemma 2.2(a) implies $\lambda'=\lambda$. We thus satisfy the hypothesis of Lemma 4.3 (with the roles of M' and M'' reversed), and so $M=I\otimes M''$ where M'' is $\mathbb{E}_8$ or $\mathbb{E}_8^*$. These M fall into Theorem 3.2(i).

5.2 $q=8$ for $3\nmid p$ and $\|\mathcal{R}\|=6$

By Lemma 4.1, $p\equiv_3 1$ here with $\mathcal{J}_L,\mathcal{J}_R\in\{\langle JK\rangle,\langle JK^2\rangle\}$ and $\mathcal{R}_L=\mathcal{J}_L\rho\cup\mathcal{J}_L\rho''$, $\mathcal{R}_R=\mathcal{J}_R\rho\cup\mathcal{J}_R\rho''$. As in Section 4.3, we can assume without loss of generality that $\mathcal{J}_L=\mathcal{J}_R=\langle JK\rangle$, say. The argument here is a simplified version of the Section 5.1 proof (e.g. Lemma 2.3(b) applies to the first component). The resulting M lie in Theorem 3.2(iii).

5.3 $q=8$ for $3\mid p$ and $\|\mathcal{R}\|=6$

Lemma 4.1 gives $\mathcal{R}_L=\mathcal{R}_R=\{\langle J1\rangle\rho\rho,\langle J1\rangle\rho\rho''\}$. This argument follows that of Sections 4.4 and especially 5.1, and the resulting M's fall into Theorem 3.2(i). Section 5.1 uses Lemma 2.2 to focus on $(2,1)\in\Phi_3^p$. This is replaced here with $(2,2),(4,1)\in\Phi_3^p$:

Lemma 5.1 *Suppose 3 divides $n>3$. Consider any $\mu,\nu\in\Phi_3^n$ with $t(\mu)\equiv_3 t(\nu)\equiv_3 0$.*

(a) $\langle J\rangle\mu=\langle J\rangle\nu$ iff both $S^{(n)}_{(2,2)\mu}/S^{(n)}_{\rho\mu}=S^{(n)}_{(2,2)\nu}/S^{(n)}_{\rho\nu}$ and $S^{(n)}_{(4,1)\mu}/S^{(n)}_{\rho\mu}=S^{(n)}_{(4,1)\nu}/S^{(n)}_{\rho\nu}$.

(b) Suppose $n>12$. Provided $\mu\notin\langle C,J\rangle(4,1)\cup\langle J\rangle(2,2)\cup\langle J\rangle\rho$,

$$\mathcal{D}^{(n)}\mu>\mathcal{D}^{(n)}(4,1)>\mathcal{D}^{(n)}(2,2)>1 \tag{5.3}$$

(if $n<12$ replace $\mathcal{D}^{(n)}(4,1)>\mathcal{D}^{(n)}(2,2)$ here with $\mathcal{D}^{(n)}(2,2)>\mathcal{D}^{(n)}(4,1)$, while if $n=12$ replace that middle inequality with an equality).

Proof of Lemma. The starting point for proving part (a) is the identity

$$\chi_\lambda(\mu):=S^{(n)}_{\lambda\mu}/S^{(n)}_{\rho\mu}=ch_{\lambda-\rho}(-2\pi\mathrm{i}\,\mu/n)\ , \tag{5.4}$$

where ch_ν is the character of the A_2-module with highest weight ν (see Section 13.9 of [22] for the generalisation to all affine algebras). The A_2-characters ch_ν for $3\mid t(\nu)$ span over $\mathbb{C}$ a subring of the character ring of A_2. This subring is generated by $ch_{(1,1)},ch_{(3,0)},ch_{(0,3)}$. To see this, it suffices (by Ch.VI, Section 3.4 of [5]) to prove the analogous statement for the leading terms, namely the formal exponentials e^ν, and this is immediate.

From (5.4) this means that any $\chi_\lambda(\mu)$ is a polynomial in $\chi_{(2,2)}(\mu)$, $\chi_{(4,1)}(\mu)$, and $\chi_{(1,4)}(\mu)=(\chi_{(4,1)}(\mu))^*$. So together $\chi_{(2,2)}(\mu)=\chi_{(2,2)}(\nu)$ and $\chi_{(4,1)}(\mu)=\chi_{(4,1)}(\nu)$ imply $\chi_\lambda(\mu)=\chi_\lambda(\nu)$ for all

$\lambda \in \Phi_3^n$ with $3 \mid t(\lambda)$. Hitting this with $S^{(n)}_{J^i\kappa,\lambda}$ and summing over $i = 0, 1, 2$ and *all* $\lambda \in \Phi_3^n$ (the sum over i projects away the weights with $3 \nmid t(\lambda)$) gives

$$\sum_i \delta_{J^i\kappa,\mu}/S^{(n)}_{\rho\mu} = \sum_i \delta_{J^i\kappa,\nu}/S^{(n)}_{\rho\nu}$$

and hence $\langle J\rangle\mu = \langle J\rangle\nu$.

To prove part (b), first compare $\mathcal{D}^{(n)}(1,4)$ and $\mathcal{D}^{(n)}(2,2)$ for $n \geq 6$: we find from (2.27) that

$$\frac{\partial}{\partial n}\frac{\mathcal{D}^{(n)}(1,4)}{\mathcal{D}^{(n)}(2,2)} = \frac{\pi}{n^2}\frac{\mathcal{D}^{(n)}(1,4)}{\mathcal{D}^{(n)}(2,2)}(2c(2) - c(1) - c(5)) \ , \tag{5.5}$$

where $c(x) = x\cot(\pi x/n)$. For fixed n, $c(x)$ is concave decreasing over the interval $0 < x < n$. Hence from (5.5), $\mathcal{D}^{(n)}(1,4)/\mathcal{D}^{(n)}(2,2)$ monotonically increases with n. We verify $\mathcal{D}^{(12)}(1,4) = \mathcal{D}^{(12)}(2,2)$, so $\mathcal{D}^{(n)}(1,4) < \mathcal{D}^{(n)}(2,2)$ for $n < 12$ and $\mathcal{D}^{(n)}(1,4) > \mathcal{D}^{(n)}(2,2)$ for $n > 12$.

Call $(a,b) \in \mathbb{Z}^2$ *admissible* if $1 \leq a \leq b \leq n-a-b$ and $a \equiv_3 b$. Then for any $\lambda \in \Phi_3^n$ with $3 \mid t(\lambda)$, exactly one $C^iJ^j\lambda$ will be admissible. Thus it suffices to consider quantum-dimensions of admissible (a,b). When $(a,b) \neq (a',b')$ are both admissible,

$$a \leq a' \text{ and } b \leq b' \ \Rightarrow\ \mathcal{D}^{(n)}(a,b) < \mathcal{D}^{(n)}(a',b') \ . \tag{5.6}$$

To see this note that $\partial/\partial x(\sin(x)\,\sin(x+y)) = \sin(2x+y)$, so $\mathcal{D}^{(n)}(a,b)$ is an increasing function of a (resp. b) for fixed b (resp. a).

Now let $\mu = (a,b) \neq (1,1),(1,4),(2,2)$ be admissible. If $a = 1$ then $b \geq 7$ and $n \geq 15$, so by (5.6) we get (5.3). Similarly, if either $a = 2$ (so $b \geq 5$), or both $a \geq 3$ and $b \geq 4$, we will have $n \geq 12$ and again (5.3) will follow from (5.6). The only remaining possibility is $\mu = (3,3)$ (for $n \geq 9$). Then exactly as in (5.5), $\mathcal{D}^{(n)}(3,3)/\mathcal{D}^{(n)}(1,4)$ is an increasing function of n. Moreover, (5.6) says $\mathcal{D}^{(n)}(3,3) > \mathcal{D}^{(n)}(2,2)$, so combining this with $\mathcal{D}^{(9)}(2,2) > \mathcal{D}^{(9)}(1,4)$ we obtain $\mathcal{D}^{(n)}(3,3) > \mathcal{D}^{(n)}(1,4)$ at $n = 9$ and hence at all n. QED

5.4 $p = 12$ and $\|\mathcal{R}\| = 6$

Here q is coprime to 6 (recall Lemma 2.3(b)), and $\mathcal{R}_L = \mathcal{R}_R = \langle J1\rangle\rho\rho \cup \langle J1\rangle\rho''\rho$. Write $[\lambda] = \langle J\rangle\lambda$. By Lemma 2.1(b), the $\lambda\mu$-row of M is nonzero iff $[\lambda] \in \{[\rho],[\rho''],[(3,3)]\}$. Evaluating $MS = SM$ at $([\rho]\mu,[\rho]\rho)$ and using the facts that $\mathcal{D}^{(12)}\rho'' = 7+4\sqrt{3}$ is irrational and $\mathcal{D}^{(12)}(3,3) = 1+\mathcal{D}^{(12)}\rho''$, we obtain for each μ two possibilities: the only nonzero entries of the $[\rho]\mu$-row are either (i) $M_{[\rho]\mu,[\rho]\mu^1} = M_{[\rho]\mu,[\rho'']\mu^2} = 1$ for $\mu^1,\mu^2 \in \langle C,K\rangle\mu$, or (ii) $M_{[\rho]\mu,[(3,3)]\mu^3} = 1$ for some $\mu^3 \in \langle C,K\rangle\mu$. Call μ 'type (i)' or 'type (ii)' resp. In type (i), $\mu^1 = \mu^2$ follows from the $([\rho]\mu,[2,2]\nu)$-entries of $MS = SM$, for all ν.

Suppose first that $M_{[\rho]\,{}^1(2,1),[\rho](2,1)} = 1$ for some ${}^1(2,1) = C^aK^b(2,1)$. Hitting M with some $I \otimes C^i\mathbb{D}_q^j$, we can require ${}^1(2,1) = (2,1)$. $MS = SM$ at $([\rho]\mu,[\rho](2,1))$ gives either $\mu^1 = \mu$ or

$\mu^3 = \mu$, whichever is appropriate, but the latter violates (2.11). Therefore Lemma 4.3 implies $M = \mathbb{E}_{12} \otimes I$ lies in Theorem 3.2(i).

It thus suffices to suppose $M_{[(3,3)]\,{}^3(2,1),[\rho](2,1)} = 1$ for some ${}^3(2,1) = C^a J^b(2,1)$. We can force $a = 0$, but (2.11) says $q \equiv_3 -1$ and $b = 1$. For any μ, $MS = SM$ at $([\rho]\mu, [\rho](2,1))$ yields $S^{(q)}_{\mu^j(2,1)} = S^{(q)}_{\mu,K(2,1)}$, i.e. $\mu^j = J^{t(\mu)}\mu$, for $j = 1$ or 3, whichever is appropriate. But then (2.11) forces μ to be of type (i) if $3 \mid t(\mu)$ and type (ii) otherwise. When μ is type (i), the Galois automorphism σ_ℓ (recall (2.13)) with $\ell' = 5, \ell'' = 1$ (recall Section 4.1) now also gives $M_{[\rho'']\mu,[\rho'']\mu} = M_{[\rho'']\mu,[\rho]\mu} = 1$.

Finally, evaluate $MS = SM$ at $([(3,3)]\mu, [\rho]\nu)$ for any μ, ν: this gives

$$\begin{aligned}\textstyle\sum_{\mu'} M_{[(3,3)]\mu,[\rho]\mu'} S^{(12)}_{\rho\rho} S^{(q)}_{\mu'\nu} + \sum_{\mu''} M_{[(3,3)]\mu,[\rho'']\mu''} S^{(12)}_{\rho''\rho} S^{(q)}_{\mu''\nu} + \sum_{\mu'''} M_{[(3,3)]\mu,[(3,3)]\mu'''} S^{(12)}_{(3,3)\rho} S^{(q)}_{\mu'''\nu})\\ = S^{(12)}_{(3,3)\rho}(S^{(q)}_{K^{t(\mu)+1}\mu,\nu} + S^{(q)}_{K^{t(\mu)-1}\mu,\nu} \ .\end{aligned} \tag{5.7}$$

Now multiply by $S^{(q)*}_{\nu\gamma}$ and sum over all ν: writing $x = 7 + 4\sqrt{3}$, this becomes

$$\begin{aligned}\textstyle\sum_{\mu'} M_{[(3,3)]\mu,[\rho]\mu'} \delta_{\mu'\gamma} + x \sum_{\mu''} M_{[(3,3)]\mu,[\rho'']\mu''} \delta_{\mu''\gamma} + (1+x) \sum_{\mu'''} M_{[(3,3)]\mu,[(3,3)]\mu'''} \delta_{\mu'''\gamma}\\ = (1+x)(\delta_{K^{t(\mu)+1}\mu,\gamma} + \delta_{K^{t(\mu)-1}\mu,\gamma}) \ ,\end{aligned} \tag{5.8}$$

valid for all $\gamma \in \Phi^q_3$. Using (2.11), this is enough to see M is the exceptional $\mathbb{E}_{12,q}$.

5.5 $p = 24$ and $\|\mathcal{R}\| = 12$

Again q is coprime to 6 so Lemma 2.3(b) applies. Recall $[\lambda] = \langle J \rangle \lambda$. From Lemma 4.1 we have $\mathcal{R}_L = \mathcal{R}_R = \cup_{i=1}^4 [\rho^i]\rho$ where for $i = 1,2,3,4$ we write $\rho^i = \rho, (5,5), (7,7), (11,11)$ respectively.

The 4 numbers $S^{(24)}_{\rho,\rho^i}$ are linearly independent over $\mathbb{Q}$. This is easy to see using the Galois automorphisms σ_5, σ_7: those σ's permute the numbers $z_i := \mathrm{i}\sqrt{3}S^{(24)}_{\rho\rho^i}$ via the permutations $(12)(34), (13)(24)$ respectively, so any linear relation of the form $\sum_i r_i z_i$, $r_i \in \mathbb{Q}$, is quickly seen to be trivial. This means $MS = SM$ at $([\rho]\mu, [\rho]\rho)$, for any μ, forces there to be weights $\mu^i \in \langle C, K \rangle \mu$ such that the only nonzero entries of the $[\rho]\mu$-row are $M_{[\rho]\mu,[\rho^i]\mu^i} = 1$. That $\mu^1 = \mu^2 = \mu^3 = \mu^4$, follows from $MS = SM$ at $(\rho\mu, (2,2)\nu)$: hitting it with $S^{(q)*}_{\nu\nu'}$ and summing over ν yields

$$S^{(24)}_{\rho^1(2,2)}(\delta_{\mu^1\nu'} - \delta_{\mu^4\nu'}) = S^{(24)}_{\rho^2(2,2)}(\delta_{\mu^3\nu'} - \delta_{\mu^2\nu'}) \ .$$

As usual (hitting M if necessary by $(1 \otimes C^i \mathbb{D}^j_q)$ we can assume the only nonzero entries of the $[\rho](2,1)$-column are $M_{[\rho^i](2,1),[\rho](2,1)} = 1$. Evaluating $MS = SM$ at $([\rho]\mu, [\rho](2,1))$ and using Lemma 2.2(a) gives $\mu^1 = \mu$ for all μ. Hence by Lemma 4.3, $M = \mathbb{E}_{24} \otimes I$ lies in Theorem 3.2(i).

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